MATRIX LIE GROUPS AND THE LIE GROUP–LIE ALGEBRA CORRESPONDENCE

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Abstract. We introduce Lie theory through the special case of matrix Lie groups. In doing so, we omit the topology that is essential to a more general study of Lie theory. In section 2, we introduce the notion of a matrix Lie group along with several basic definitions and properties of matrices. Section 3 defines and details matrix exponentiation; in section 4, we discuss *tangent spaces* and the *Lie bracket*, and in section 5 their connections to Lie algebras.

1. Introduction

Sophus Lie developed Lie theory, which is centered around the concept of continuous groups — groups with a continuous operation. His original idea was to build on the work of Klein and Poincaré, who developed discrete groups in the study of modular forms. One of Lie's most important discoveries was that continuous automorphism groups, or *Lie groups*, were easier to work with when "linearized"; differentiating a Lie group action yields a linear *Lie algebra* action. We will discuss the exponential map that relates a Lie algebra to its Lie group, known as the *Lie group-Lie algebra correspondence*.

In differential geometry, differentiating a Lie group action yields a linear Lie algebra action. Lie's initial application of his theory was to differential equations, modeled after Galois theory and the study of symmetry. Lie groups also play central roles in several other branches of mathematics, including representation theory and algebraic topology. They also play a role in non-mathematical fields: representation theory is used extensively in particle physics; Lie groups and Lie algebras are often used in computer vision and finance.

2. General Linear Group

Here, we define several basic concepts and go through a few examples that will be used throughout this paper.

Definition 2.1. Denote the space of all $n \times n$ matrices with entries in ring R by $M_n(R)$. Throughout this paper, we will only consider $R = \mathbb{R}$ and $\mathbb{C}$.

Although $M_n(R)$ is a ring under matrix addition and multiplication, but it has zero divisors, making it less interesting. For a given non-invertible matrix $A \in M_n(R)$, take a vector $\vec{v} \in R^n$ with $A\vec{v} = \mathbf{0}$, and define

$$B = \begin{pmatrix} | & & | \\ v & \cdots & v \\ | & & | \end{pmatrix}$$

Then $AB = \mathbf{0}$. Let us take a look at $M_n(R)$'s multiplicative group, $GL_n(R)$, which turns out to be much more interesting.

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Definition 2.2. The *general linear group* of degree n over a ring R, denoted $GL_n(R)$, is the set of all $n \times n$ invertible matrices with entries in V.

Let us first check that $GL_n(R)$ is actually a group. Let $A, B \in GL_n(R)$, so $\det(A), \det(B) \neq 0$. Hence $\det(AB) = \det(A)\det(B) \neq 0$, and so $AB \in GL_n(R)$ (closure). Associativity follows from the (well-known) associativity of matrix multiplication. The identity matrix is the identity matrix in $M_n(R)$.Lastly,

$$1 = \det(I) = \det(A)\det(A^{-1}) \implies \det(A^{-1}) \neq 0 \implies A^{-1} \in GL_n(R),$$

so $GL_n(R)$ is indeed a group.

Now, before we move onto the definition of a matrix Lie group, we must first introduce the norm of a general matrix.

Definition 2.3. The *Hilbert-Schmidt norm*, which we will simply refer to as the *norm*, of a matrix A is

$$\|A\| = \sqrt{\sum_{i,j} a_{ij}^2}.$$

The *distance* between two matrices is given by the Euclidean distance, $d(A, B) = \|A - B\|$. If we are given a sequence of matrices (A_n), the sequence is said to *converge* to a matrix A if

$$\lim_{n\to\infty} d(A_n, A) = 0.$$

It is easy to see that A_n converges entry-wise to A — the norm is always nonnegative, and it is 0 if and only if the matrix is $\mathbf{0}$.

Remark. Other norms, like the "taxicab norm" $\sum_{i,j} |a_{ij}|$ and the "max norm" $\max_{i,j} |a_{ij}|$, also work for this notion of convergence. We will not exploit any specific properties of the specific norms in this paper.

We now prove a few properties of the norm that will allow us to further characterize exp.

Proposition 2.4. *The following are properties of $\|\cdot\|$, where $A, B \in M_n(\mathbb{C})$:*

(a) $\lambda \|A\| = \|\lambda\| A$ for $\lambda \in \mathbb{C}$.
(b) $\|A + B\| \leq \|A\| + \|B\|$.
(c) $\|AB\| \leq \|A\| \|B\|$.

Proof. (a) Follows from moving λ outside of the sum $\sum_{i,j} (\lambda a_{ij})^2$.

(b) "Reshape" $A + B, A$, and B like vectors under the same (ℓ_2) norm. Since the triangle inequality holds for vectors, it holds for matrices. Likewise, (c) follows from the Cauchy-Schwarz Inequality. ■

Now that we know what convergence means, we are ready to start looking at matrix Lie groups.

Definition 2.5. A *matrix Lie group* is a closed subgroup G of $GL_n(R)$. By closed, we mean that for any sequence (A_n) of matrices in G converging to a matrix A, then either $A \in G$ or $A \notin GL_n(R)$.

Remark. The idea of "closedness" is the same as in topology. In fact, our metric d allows us to speak of topology on $GL_n(R)$. However, we will not go into topological details in this paper.

It is obvious that the general linear group itself is a matrix Lie group. Below, we will investigate a few well-known matrix Lie groups.

Example 2.6. Define the *orthogonal group*

$$O(n) = \{A \in M_n(\mathbb{R}) : A^\intercal A = I\}$$

The *special orthogonal group* $SO(n) \leq O(n)$ contains only the matrices of positive determinant. Geometrically, $O(n)$ is the group of isometries of $\mathbb{R}^n$ that have a fixed point; its operation is the composition of transformations, or equivalently, the multiplication of matrices.

Now we can check that this is a group. Let A, B be orthogonal. Then

$$(AB)^\intercal AB = B^\intercal A^\intercal AB = B^\intercal B = I \implies AB \in O(n)$$
$$(A^{-1})^\intercal A^{-1} = (A^\intercal)^{-1}A^{-1} = (AA^\intercal)^{-1} = I^{-1} = I \implies A^{-1} \in O(n)$$

To show that it is a matrix Lie group, observe

$$1 = \det(I) = \det(A^\intercal A) = \det(A^\intercal)\det(A) = \det(A)^2,$$

so $\det(A) = \pm 1$. If $A_n \to A$, then $\det(A_n) \to \det(A)$, so $\det(A_n)$ is eventually constant.

Notice that SO is in fact a subgroup of O, as it is closed under multiplication and inversion and is a matrix Lie group for the same reasons as above.

We will not discuss the following example in much detail throughout the paper; however, it is worth noting that many $SU(n)$'sappear frequently in both mathematics and physics. For instance, $SU(2)$ is isomorphic to the group of quaternions with norm 1, and it is diffeomorphic to S^3. It is often used to model electromagnetism and the weak nuclear force. $SU(3)$ is often used in quantum chromodynamics. See [2] for a more detailed exposition on the role of the special unitary group in particle physics.

Example 2.7. Let $A \in GL_n(\mathbb{C})$, and define $A^* = (\overline{a_{ji}})$ to be the *adjoint* of A, where $\overline{\cdot}$ denotes complex conjugation. The *unitary group* $U(n) = \{A : A^*A = I\}$ is a subgroup of $GL_n(\mathbb{C})$. Also define the subgroup $SU(n) \leq U(n)$ consisting of A with determinant 1, called the *special unitary group*.

First we show that $U(n)$ is indeed a subgroup of $GL_n(\mathbb{C})$. Let $A, B \in U(n)$. Then

$$(AB)^*(AB) = B^*A^*AB = B^*B = I \implies AB \in U(n)$$
$$(A^{-1})^*(A^{-1}) = (A^*)^*(A^*) = AA^* = AA^{-1} = I \implies A^{-1} \in U(n)$$

Also notice that since

$$1 = \det(I) = \det(A^*A) = \det\left(\overline{A^\intercal}\right)\det(A) = \overline{\det(A^\intercal)}\det(A) = \overline{\det(A)}\det(A) = |\det(A)|^2,$$

we must have $|\det(A)| = 1$. Consider A_n, a sequence in $U(n)$. Their determinants all lie on the unit circle; so does the limit of their determinants. Hence $A \in U(n)$, and the unitary group is a matrix Lie group. The same can be said of $SU(n)$.

3. Matrix Exponential

The exponential function for matrices is instrumental in the study of Lie groups. We will see applications of this analog of the exponential function in $\mathbb{C}$ in future sections.

Definition 3.1. If $A \in M_n(\mathbb{C})$, we write

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

For example,

$$e^{\mathbf{0}} = \sum_{k=0}^{\infty} \frac{\mathbf{0}^k}{k!}.$$

All terms with $k > 0$ vanish, leaving only $\frac{\mathbf{0}^0}{0!} = I$. Thus, $e^{\mathbf{0}} = I$, as one might expect.

We will now state and prove two fundamental elementary properties of exp, which we will use implicitly throughout this paper.

Proposition 3.2. exp *is defined (i.e. the sum converges) for all* A, *and* exp *is a continuous map.*

Proof. For the first part, we have

$$0 \leq \sum_{k=0}^{\infty} \left\| \frac{A^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = e^{\|A\|}$$

is a bounded sum of nonnegative reals.

For the second part, let r be a positive real number and $B(r) = \{A \in M_n(\mathbb{R}) : |A| \leq r\}$. Let $M_n = r/n!$. Then

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \frac{r}{n!} = e^r$$

and

$$\left\| \frac{A^n}{n!} \right\| = \frac{\|A^n\|}{n!} \leq \frac{\|A\|^n}{n!} = \frac{r^n}{n!}$$

for each $A \in B(r)$. Then $\exp(A)$ is continuous on $B(r)$ by the Weierstraß M-test. For each $X \in M_n(\mathbb{R})$, there exists an r such that $X \in B(r)$, hence exp is continuous on $M_n(\mathbb{R})$. ∎

Now we will discuss several important properties of the exponential function. The following proposition allows us to efficiently deal with exponentiated conjugates.

Proposition 3.3. *If* $A \in M_n(\mathbb{C}), B \in GL_n(\mathbb{C})$, *then*

$$e^{BAB^{-1}} = Be^AB^{-1}.$$

Proof. It is well-known that

$$(BAB^{-1})^n = BAB^{-1}BAB^{-1} \cdots BAB^{-1} = BAA \cdots AB^{-1} = BA^nB^{-1},$$

from which we can substitute to obtain the desired result:

$$e^{BAB^{-1}} = \sum_{n=0}^{\infty} \frac{(BAB^{-1})^n}{n!} = \sum_{n=0}^{\infty} \frac{BA^nB^{-1}}{n!} = B\left(\sum_{n=0}^{\infty} \frac{A^n}{n!}\right)B^{-1} = Be^AB^{-1}.$$ ∎

The proposition works particularly well in the case of diagonalizable matrices. If the eigenvectors $v_1, v_2, \ldots, v_n$ (with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$) of A are linearly independent, define

$$C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}, \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Thus, $A = CDC^{-1}$, and thus

$$e^A = Ce^DC^{-1} = C \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} C^{-1}.$$

The following related theorem allows us to compute the determinant of the exponential (although not the exponential itself) very efficiently.

Theorem 3.4. *For $A \in M_n(\mathbb{C}), \det(e^A) = e^{\mathrm{Tr}(A)}$.*

Proof. Apply Jordan's Theorem. Set $A = PQP^{-1}$, where Q is upper triangular. Then

$$A^n = (PQP^{-1})^n = PQ^nP^{-1}.$$

Thus,

$$\begin{aligned}
\det(e^A) &= \det\left[\sum_{i=0}^{\infty} \frac{PQ^iP^{-1}}{i!}\right] \\
&= \det\left[P\left(\sum_{i=0}^{\infty} \frac{Q^i}{i!}\right)P^{-1}\right] \\
&= \det(P)\det(e^Q)\det(P^{-1}) \\
&= \det(e^Q).
\end{aligned}$$

However, since

$$\mathrm{Tr}(Q) = \mathrm{Tr}(PQP^{-1}) = \mathrm{Tr}(A),$$

it remains to show that $\det(e^Q) = e^{\mathrm{Tr}(Q)}$.

To do so, observe that Q^j is upper triangular with $(Q^j)_{ii} = (Q_{ii})^j$. It follows that e^Q is upper triangular with $(e^Q)_{ii} = e^{Q_{ii}}$. Since the determinant of an upper triangular matrix is the product of the entries on its diagonal, we have:

$$\det(e^Q) = \prod_{i=1}^{n} e^{Q_{ii}} = e^{\sum_{i=1}^{n} Q_{ii}} = e^{\mathrm{Tr}(Q)},$$

as desired. ∎

We also enumerate a few important properties of the exponential function:

Theorem 3.5. *Let $A, B \in M_n(\mathbb{C})$ that commute, i.e. $AB = BA$. Then:*

$$e^{A+B} = e^Ae^B = e^Be^A.$$

As a corollary, e^A is invertible and $\left(e^A\right)^{-1} = e^{-A}$.

Proof. Expand:

$$e^A e^B = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{A^i}{i!}\cdot\frac{B^j}{j!} \tag{1}$$

$$= \sum_{s=0}^{\infty}\sum_{i=0}^{s} \frac{A^i}{i!}\cdot\frac{B^{s-i}}{(s-i)!} \tag{2}$$

$$= \sum_{s=0}^{\infty}\frac{1}{s!}\sum_{i=0}^{s}\binom{s}{i}A^iB^{s-i} \tag{3}$$

$$= \sum_{s=0}^{\infty}\frac{(A+B)^s}{s!} \tag{4}$$

$$= e^{A+B}, \tag{5}$$

where (2) represents first summing over $s = i + j$ then iterating over $0 \leq i \leq s$ and (4) utilizes the commutativity of A, B and the Binomial Theorem.

As for the corollary, substituting $B = -A$ gives

$$e^A e^{-A} = e^{A-A} = I,$$

as desired. ■

Since Theorem 3.5 only holds when the matrices commute, it may be helpful to find a more general way of finding e^{A+B}. The following theorem does so, but its formal proof is beyond the scope of this paper. We provide a brief sketch; see [1] for details on the enumerated facts. Theorem 6.5 gives another method of generalization.

Theorem 3.6 (Lie Product Formula)**.** *For all* $A, B \in M_n(\mathbb{C})$*, we have*

$$e^{A+B} = \lim_{p\to\infty}\left(e^{A/p}e^{B/p}\right)^p.$$

Proof Sketch. Define the matrix logarithm on $M_n(\mathbb{C})$ the same way as the logarithm is defined on $\mathbb{C}$,

$$\log A = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{(A-I)^i}{i}.$$

We state without proof the following properties of the logarithm:

(1) When $\|A - I\| < 1$, this converges and $e^{\log A} = A$.
(2) For all $A \in M_n(\mathbb{C})$ with $\|A\| < \frac{1}{2}$,

$$\log(A + I) = A + O(\|A\|^2).$$

Thus, as $p \to \infty, e^{A/p}e^{B/p}$ approaches $I \cdot I = I$, and hence it is in the domain of log.

$$\begin{aligned}
\log(e^{A/p}e^{B/p}) &= \log\left(1 + \frac{A}{p} + \frac{B}{p} + O\left(\frac{1}{p^2}\right)\right) \\
&= \frac{A}{p} + \frac{B}{p} + O\left(\frac{1}{p^2}\right) \\
e^{A/p}e^{B/p} &= \exp\left(\frac{A}{p} + \frac{B}{p} + O\left(\frac{1}{p^2}\right)\right) \\
\left(e^{A/p}e^{B/p}\right)^p &= \exp\left(A + B + O\left(\frac{1}{p}\right)\right),
\end{aligned}$$

from which the desired result follows when we take $p \to \infty$. ■

The matrix exponential satisfies yet another very useful property: since we have dedicated quite some attention to multiplication, inverses, and determinants, let us now turn our attention to differentiation. Before we do so, let us first define a path.

Definition 3.7. A *path* in G is a map $\gamma : \mathbb{R} \to G$ that sends a variable t to a matrix $A(t) = (a_{ij})$. In a *smooth* path, each entry is differentiable in t, in which case we write $A'(t) = (a'_{ij})$.

Proposition 3.8. *Let $A \in M_n(\mathbb{C})$. Then e^{tA} is smooth, and*

$$\frac{d}{dt}e^{tA} = Ae^{tA}.$$

Notably, the derivative at 0 is A.

Proof. Since power series are differentiable within their radius of convergence, we may differentiate e^{tA} term-by-term.

$$\begin{aligned}
\frac{d}{dt}\left(\sum_{k=0}^{\infty}\frac{A^k t^k}{k!}\right) &= \sum_{k=0}^{\infty}\frac{d}{dt}\left(\frac{A^k t^k}{k!}\right) \\
&= \sum_{k=0}^{\infty}\frac{A^k t^{k-1}}{(k-1)!} \\
&= A\sum_{k=0}^{\infty}\frac{A^k t^k}{k!} \\
&= Ae^{tA}.
\end{aligned}$$

■

4. Tangent Spaces

We begin with a definition.

Definition 4.1. Let G be a matrix Lie group. A matrix X is in the *tangent space at the identity*, denoted $T(G)$, if and only if there exists a smooth path γ in $M_n(\mathbb{C})$ satisfying $\gamma(0) = I$ and $X = \gamma'(0)$.

Tangent spaces are vector spaces. To see this, we must show that it is closed under addition and satisfies the standard condition for scalar multiplication.

For addition, suppose that A, B are paths in G and $A'(0), B'(0) \in T(G)$. Consider the path AB. $A(0)B(0) = II = I$, and its derivative at 0 is

$$A'(0)I + IB'(0) = A'(0) + B'(0).$$

For scalar multiplication, take the path A with $A'(0) \in T(G)$. Now consider the path $A(\lambda t)$. This also passes through $(0, I)$, and its derivative at 0 is

$$\lambda A'(\lambda \cdot 0) = \lambda A'(0).$$

The importance of the exponential function is that it maps $T(G)$ to G for a matrix group G. A simple "almost-example" of this is the unit circle in $\mathbb{C}$. The tangent space at the identity can be found to be $1 + i\theta$; the exponential function is $e \cdot e^{i\theta}$ is a circle centered at 0 with radius e, which is isomorphic to the unit circle.

Now, let us investigate a more generalizable example: $T(SO(n))$.

Definition 4.2. A matrix $A \in M_n(\mathbb{R})$ is *skew-symmetric* if $A + A^\intercal = \mathbf{0}$. Thus, A has zeros along its diagonal, and $a_{ij} = -a_{ji}$. We denote the set of skew-symmetric matrices as $so(n)$.

Proposition 4.3. *Let $A \in so(n)$. Then $f(t) = e^{tA} \in SO(n)$.*

Proof. We are given that $A^\intercal = -A$. Therefore,

$$AA^\intercal = A(-A) = (-A)A = A^\intercal A.$$

By Theorem 3.5,

$$e^A (e^A)^\intercal = e^A e^{A^\intercal} = e^0 = I \implies \exp(A) \in O(n).$$

The same holds for tA in place of A, so $f(t) \in O(n)$.

Now, note that $|\det(f(t))| = 1$. At $t = 0$, $\det(e^0) = \det(I) = 1$, so by continuity, $\det(f(t)) = 1$. The result follows. ■

Proposition 4.4. $T(SO(n)) = so(n)$.

Proof. First, we show that $T(SO(n)) \subseteq so(n)$. Let $X \in T(SO(n))$, so there is a path γ with $\gamma(0) = I$ and $\gamma'(0) = X$. Note that since γ is a path through $SO(n)$, for all $t \in \mathbb{R}$, we have

$$(\gamma(t))^\intercal(\gamma(t)) = I.$$

Differentiating, Differentiating,

$$(\gamma(t))^\intercal(\gamma'(t)) + (\gamma'(t))^\intercal(\gamma(t)) = 0.$$

Substituting $t = 0$, we obtain

$$\gamma'(0) + \gamma'(0)^\intercal = 0,$$

and thus $X = \gamma'(0) \in so(n)$.

Now, we prove the other direction, $T(SO(n)) \supseteq so(n)$. Consider a matrix $X \in so(n)$. Define a path $\gamma(t) := e^{tX}$ (in $SO(n)$ by Proposition 4.3). Then:

$$\frac{d}{dt} e^{tX} = X e^{tX};$$

substituting $t = 0$ yields $\gamma'(0) = X$ as desired. ■

5. Lie Algebras

In the previous section, we showed that the tangent space of $SO(n)$ is precisely the skew-symmetric $n \times n$ matrices, $so(n)$. In this section, we will demonstrate how this connects to the larger study of Lie theory.

Definition 5.1. A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a bivariate *bracket* operator $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is (for all $A, B, C \in \mathfrak{g}$ and constant λ):

(1) *bilinear*, i.e.

$$[A+B, C] = [A, C] + [B, C] \qquad [\lambda A, B] = \lambda[A, B]$$
$$[A, B+C] = [A, B] + [A, C] \qquad [A, \lambda B] = \lambda[A, B],$$

(2) *anticommutative*, i.e. $[A, B] = -[B, A]$,

(3) and satisfies the *Jacobi identity*, i.e.

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

Perhaps the simplest example of a Lie algebra is the vector space $\mathbb{R}^3$ equipped with the cross product $\times$ as its bivariate operation. The first two properties of the cross product are well known; the last follows from Lagrange's formula on triple products,

$$\sum_{\text{cyc}} A \times (B \times C) = \sum_{\text{cyc}} B(A \cdot C) - C(A \cdot B) = \mathbf{0}.$$

For our purposes, however, the following choice of $[\cdot,\cdot]$ is most convenient:

Definition 5.2. Let G be a matrix Lie group. Then for any $A, B \in T(G)$, the *Lie bracket* is

$$[A, B] = AB - BA.$$

Bilinearity and anticommutativity are obvious. As for the Jacobi identity, observe that

$$\begin{aligned}\sum_{\text{cyc}} [A, [B, C]] &= \sum_{\text{cyc}} A[B, C] - [B, C]A \\ &= \sum_{\text{cyc}} A(BC - CB) - (BC - CB)A \\ &= \sum_{\text{cyc}} ABC - ACB - BCA + CBA.\end{aligned}$$

It is not hard to check that each of the 6 permutations appears twice — once positively and once negatively, and hence they total 0. Thus, this selection of a bracket is a "valid" Lie bracket. (Note that choices that we might expect to work, like matrix addition and multiplication, fail to satisfy the Jacobi identity.)

We remark that if G is abelian, then $[A, B] = [B, A] = \mathbf{0}$ for any input. For this reason, our "almost-example" with the unit circle in Section 4 fails to generalize to non-abelian groups.

Recall that in the last section, we proved that $T(G)$ is a vector space. It turns out that we can apply what we've said about the Lie bracket to $T(G)$, thus connecting each Lie group with a corresponding Lie algebra. To do so, we show that $T(G)$ is closed under the Lie bracket.

Proposition 5.3. *Let $A, B \in T(G)$. Then $[A, B] \in T(G)$.*

Proof. Define $\gamma(t) = e^{tA}Be^{-tA}$. The derivative is, by Proposition 3.8,

$$e^{tA}(Be^{-tA})' + (e^{tA})'(Be^{-tA}) = e^{tA}(-BAe^{-tA}) + Ae^{tA}(Be^{-tA}),$$

which is $AB - BA$ at 0. Thus, by the closure of $T(G)$,

$$AB - BA = \lim_{h\to 0} \frac{e^{tA}Be^{-tA} - B}{h}$$

is a member of $T(G)$, as desired. ■

With this in mind, we are now ready to correspond Lie groups and Lie algebras; this definition should be obvious based on our results for $SO(n)$ and $\mathfrak{so}(n)$.

Definition 5.4. Denote the Lie algebra of a Lie group G by $\mathfrak{g}$; it is $T(G)$ equipped with the Lie bracket as its bilinear operator.

6. Lie Group and Lie Algebra Homomorphisms

Now that we have a solid understanding of the notion of Lie algebras, we are able to meaningfully discuss Lie group and Lie algebra homomorphisms. In this paper, we only consider continuous homomorphisms (it turns out to be quite difficult to construct discontinuous ones).

Theorem 6.1. *Let G and H be matrix Lie groups. Suppose $\Phi : G \to H$ is a homomorphism, then there is a homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ with*

$$\Phi(e^A) = e^{\varphi(A)}.$$

Proof. The following proof can be found in [1]. Select Z such that

$$\Phi(e^{tA}) = e^{tZ} \tag{6}$$

for all $t \in \mathbb{R}$. We claim that $\varphi(A) := Z$ is a homomorphism. It is clear that $\varphi(tA) = t\varphi(A)$. By the Lie product formula and the continuity of Φ, we have

$$\begin{aligned}
e^{t\varphi(A+B)} &= \Phi(e^{t(A+B)}) \\
&= \Phi\left(\lim_{p\to\infty} \left(e^{tA/p}e^{tB/p}\right)^p\right) \\
&= \lim_{p\to\infty} \left(\Phi(e^{tA/p})\Phi(e^{tB/p})\right)^p \\
&= \lim_{p\to\infty} \left(e^{t\varphi(A)/p}e^{t\varphi(B)/p}\right)^p \\
&= e^{t(\varphi(A)+\varphi(B))},
\end{aligned}$$

and differentiating at 0 yields $\varphi(A + B) = \varphi(A) + \varphi(B)$ by Proposition 3.8. ■

Definition 6.2. A *Lie subalgebra* is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ that is closed under the Lie bracket. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is an *ideal* if

$$A \in \mathfrak{g}, B \in \mathfrak{h} \implies [A, B] \in \mathfrak{h}.$$

Note that there are subalgebras that are not ideals:

Example 6.3. Consider the Lie algebra of diagonal 2×2 matrices $\mathfrak{d}(2) \subset \mathfrak{gl}(2)$. Their bracket is

$$\begin{aligned}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} ax & by \\ cx & dy \end{pmatrix} - \begin{pmatrix} ax & bx \\ cy & dy \end{pmatrix} \\ &= \begin{pmatrix} 0 & b(y-x) \\ c(x-y) & 0 \end{pmatrix},\end{aligned}$$

which is of course not diagonal. However, it is easy to see that the bracket of two diagonal matrices is also diagonal, as in this case we would have $b = c = 0$. Thus $\mathfrak{d}(2)$ is a subalgebra, but not an ideal.

Proposition 6.4. *If H is a normal subgroup of G, then $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.*

Proof. In our proof of Proposition 5.3, we saw that

$$[A, B] = \frac{d}{dt} e^{tA} B e^{-tA}\bigg|_{t=0}$$

Since $H \lhd G$, we have $e^{tA} B e^{-tA} \in H$, and hence the result follows. ∎

Now, we saw in Theorem 3.5 that $e^{A+B} = e^A e^B$ where A and B commute. Suppose that we instead try to solve $e^A e^B = e^C$ for C in the case that A and B do not necessarily commute. Then $C = \log(e^A e^B)$; expansion shows that

$$\begin{aligned} C &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left[\left(\sum_{i=0}^{\infty} \frac{A^i}{i!}\right)\left(\sum_{j=0}^{\infty} \frac{B^j}{j!}\right) - 1\right]^k \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\sum_{i+j\geq 1} \frac{A^i B^j}{i!j!}\right)^k \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{i_\ell + j_\ell \geq 1\, \forall \ell \in [k]} \frac{A^{i_1} B^{j_1} \cdots A^{i_\ell} B^{j_\ell}}{i_1! j_1! \cdots i_\ell! j_\ell!}. \end{aligned}$$

A more general result, known as *Dynkin's Formula*, can be found by taking the derivative of the exponential function, manipulating it, and integrating. We omit the details of its proof. For simplicity, we will denote by ad_A the linear transformation mapping $B \mapsto [A, B]$. Thus, $(\mathrm{ad}_A)^2(B) = [A, [A, B]]$.

Theorem 6.5 (Dynkin's Formula)**.**

$$C = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{\forall 1 \leq \ell \leq k, \ell + j_\ell \geq 1} \frac{\left((\mathrm{ad}_A)^{i_1} (\mathrm{ad}_B)^{j_1} \cdots (\mathrm{ad}_A)^{i_k} (\mathrm{ad}_B)^{j_k}\right)(I)}{(i_1 + j_1 + \cdots + i_k + j_k)(i_1! j_1! \cdots i_k! j_k!)}.$$

The first few terms are well-known and are given by

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [B, [B, A]]) - \frac{1}{24}[B, [A, [A, B]]] + \cdots$$

Note that When A, B commute, all of the terms involving brackets vanish, and we are left with only the $A + B$ term. Thus, the statement agrees with Theorem 3.5.

Also note that the sum in Dynkin's Formula shows that C may be expressed in terms of nested Lie brackets with rational coefficients. We conclude by stating the Baker–Campbell-Hausdorff formula, whose proof is beyond the scope of the paper. See Chapter 5 of [1] for more details.

Theorem 6.6 (BCH). *Define*

$$g(Z) := \frac{Z \log Z}{Z-1} = 1 - \sum_{n=1}^{\infty} \frac{(1-Z)^n}{n(n+1)}$$

wherever it is defined. Then:

$$\log(e^A e^B) = A + \left(\int_0^1 g\left(e^{\operatorname{ad}_A} e^{t \operatorname{ad}_B}\right) dt \right)(B),$$

where the integrand is a linear operator on $M_n(\mathbb{C})$.

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References

[1] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate Texts in Mathematics. Springer, 2003.

[2] Francis Halzen and Alan D. Martin. *Quarks and leptons: An introductory course in modern particle physics*. Wiley, 1984.

[3] Maxwell Levine. *$GL_n(\mathbb{R})$ as a Lie group*. `http://www.math.uchicago.edu/~may/VIGRE/VIGRE2009/REUPapers/Levine.pdf`, 2009.

[4] Max Lloyd. *Matrices Lie: An introduction to matrix Lie groups and matrix Lie algebras*. Whitman College, `https://www.whitman.edu/documents/Academics/Mathematics/2015/Final%20Project%20-%20Lloyd.pdf`, 2015.

[5] Zuoqin Wang. *Lecture 12: Lie's Fundamental Theorems*. `http://staff.ustc.edu.cn/~wangzuoq/Courses/13F-Lie/Notes/Lec%2012.pdf`, 2012.

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