LIFTING PROPERTIES AND CLASSIFICATION OF COVERING SPACES

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AUGUST 2021

Abstract. We study the fundamental group in general topological spaces and its relationship to covering spaces. Great focus is placed on lifting paths and homotopies in covering spaces and their relationship to the induced homomorphism. Our discussion concludes with the universal cover — proving existence and uniqueness in a direct constructing and highlighting its utility through applications. We conclude with a discussion of the relationship between covering spaces and the Galois correspondence.

1. Introduction

The crux of topology is to determine which spaces are topologically equivalent. The most clear method of assessment is via explicit construction a homeomorphism, i.e. a continuous bijection with a continuous inverse, between two given spaces. However, doing so in general is a difficult task; throughout this paper, we explore the implications of the *fundamental group*: a homeomorphism invariant that, unlike the integer Euler characteristic $\chi(S)$ and boolean orientability for surfaces, associates topological space S with a group, denoted $\pi_1(S)$. In particular, homeomorphic topological spaces have isomorphic fundamental groups, providing an algebraic structure to topology.

We investigate the applications of the fundamental group to *covering spaces*, characterized by a continuous function from the covering space C to the base space X such that each point in X has an open neighborhood evenly covered by the map. Most of the results in this paper are not from considering the fundamental group in isolation, but rather determining relationships between different basepoints and covering spaces. The main thrust of this paper — characterizing the *universal cover* — provides that the covering is unique up to isomorphism for "sufficiently good" topological spaces. Aside from their deep relationship to the fundamental group, covering spaces also play a significant role in homotopy theory, harmonic analysis, Riemannian geometry, and differential topology.

2. Background

In this section, we introduce several basic concepts from algebraic topology that will be used throughout this paper. In this section, we will generally skim over proofs of stated properties; it is fairly straightforward to complete the sketches. Nevertheless, many of these properties are interesting, and some notation may be unfamiliar or unconventional. For a more detailed exposition, see [BBRS16].

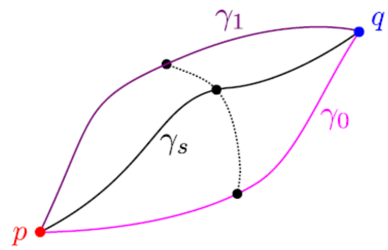


FIGURE 1. A visual representation of a homotopy between γ_0 and γ_1. γ_s is one frame in the animation, and the specific mapping is shown by the dotted line. [BBRS16]

Definition 2.1. A *continuous path* on X between two points $p, q \in S$ is a continuous mapping $\gamma : [0,1] \to X$ satisfying $\gamma(0) = p$ and $\gamma(1) = q$.[1] We call p the *starting point* and q the *ending point*. If $p = q$, we call γ a *loop based at* p.

As is customary, we will be a bit sloppy with notation, referring to γ as both the geometric path and its formal definition as a function. Visually, one can interpret a path as the result of drawing some curve on X without lifting their pen over a finite time period.

Definition 2.2. Let $f, g : X \to Y$ be continuous. We say that f and g are *homotopic*, denoted $f \sim g$, if there exists a continuous map $H : [0,1] \times X \to Y$, called the *homotopy*, satisfying $H(0,x) \equiv f(x)$ and $H(1,x) \equiv g(x)$. We say that f and g are homotopic *relative to* A, denoted $f \sim g \operatorname{rel} A$, if $H(t,a) = f(a) = g(a)$ for all $a \in A$.

The high-level idea of this definition is that one may continuously deform f to g according to H. Intuitively H serves as an animation: plotting $\gamma_s : [0,1] \to Y$ defined by $\gamma_s(x) := H(s,x)$ as s varies on $[0,1]$ interpolates between γ_0 and γ_1. The explicit interpolation mapping $\gamma_0(x)$ to $\gamma_1(x)$ is given by $\gamma_x(s) := H(s,x)$, with $\gamma_x(0) = \gamma_0(x)$ and $\gamma_x(1) = \gamma_1(x)$. "Relative" serves more as convenient notation than as a meaningful concept; in Figure 1, for instance, we can write $\gamma_0 \sim \gamma_1 \operatorname{rel} \{0,1\}$ because the paths share points p and q at times 0 and 1 respectively.

Example 2.3. Consider the topological spaces given by $X = \{0\}$ and $Y = [0,1]$. We define $H : [0,1] \times X \to Y$ by $H(t,x) = tx$, which is clearly continuous. Since $H(0,x) \equiv 0$ and $H(1,x) \equiv x$, it follows that H is a homotopy mapping id_X to id_Y where id denotes the identity map. $\square$

As we will see throughout this section, paths and homotopies behave nicely together. In particular, we introduce a group structure involving paths modulo a set of homotopy "classes." To get here, we need to (1) define the group operator and (2) determine the homotopy classes.

Let us begin by defining the multiplication. To do so, we will first define a multiplication (concatenation) operator $*$ between paths, then extend it naturally to classes of paths, from which we can accomplish both (1) and (2).

[1] It is common to define paths on intervals $[a,b] \subset \mathbb{R}$ rather than defining it only on $[0,1]$. However, the notion of such paths is identical; they are merely parameterized differently.

Definition 2.4. Let γ and γ' be paths in x satisfying $\gamma(1) = \gamma'(0)$. Then

$$\gamma * \gamma'(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \gamma(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

Essentially, concatenating two paths means traveling on one path twice as fast as usual, then proceeding along the other path twice as fast as usual. This is continuous if and only if the endpoints agree, hence the $\gamma(1) = \gamma'(0)$ condition. As we will see, this definition works well with homotopies:

Proposition 2.5. *Let $\gamma_1, \gamma_1', \gamma_2, \gamma_2'$ be paths in X such that the endpoints of γ_i and γ_i' agree. Further assume that $\gamma_1 \sim \gamma_2$ and $\gamma_1' \sim \gamma_2'$. Then $\gamma_1 * \gamma_1' \sim \gamma_2 * \gamma_2'$.*

Proof Sketch. The idea is to concatenate the respective homotopies F mapping γ_1 to γ_2 and F' mapping γ_1' to γ_2'. Specifically, we define

$$H(s,t) = \begin{cases} F(s, 2t) & t \in [0, \frac{1}{2}] \\ F'(s, 2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

From here, it is straightforward to prove that this is continuous and satisfies the needed properties of a homotopy. ■

In summary, we can use $\sim$ as an equivalence relation, which satisfies useful properties under $*$:

Proposition 2.6. *$\sim$ is an equivalence relation. In particular, $\sim$ gives rise to an equivalence class $[\gamma]$ of γ, which we call the* homotopy class*.*

Proof Sketch. Proving that $\sim$ is reflexive and symmetric is trivial. To prove that it is symmetric, one may draw inspiration from Definition 2.4, with concatenation taken with respect to s rather than x. ■

The previous several propositions enable us to define multiplication on homotopy classes in the following well-defined manner:

$$[\gamma] \cdot [\gamma'] = [\gamma * \gamma'].$$

Thus, as previously alluded to, all of this comes together:

Definition 2.7. Let X be a topological space and $x \in X$. The *fundamental group of X with basepoint x* is defined as

$$\pi_1(X, x) = \{[\gamma] : \gamma \text{ is a loop based at } x\}.$$

The first thing to note about the fundamental group is that our assertion is correct, that it is a group under multiplication of homotopy classes.

Proof Sketch.

- $\cdot$ is associative. To do this, one must construct a homotopy between $\gamma_1 * (\gamma_2 * \gamma_3)$ and $(\gamma_1 * \gamma_2) * \gamma_3$. A reparameterization is needed, as these concatenations traverse the γ_i's at different speeds. One may explicitly define it or, in fact, show more generally that any reparameterization (i.e. $f \circ \varphi$, where *varphi* $: [0,1] \to [0,1]$ is a continuous map satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$) preserves the homotopy class. We omit this proof, but it is fairly straightforward.
- We claim that the identify function defined by the constant path $e(t) := x$ works; construct a desirable homotopy based on a suitable reparameterization.
- For the inverse, define $\bar{\gamma}$ by traversing γ in reverse: $\bar{\gamma}(t) := \gamma(1-t)$. Construct a reparameterization to show that $\gamma * \bar{\gamma} \sim e$. ■

Given this baseline, we are able to deduce useful conclusions regarding the fundamental group. In particular, observe the following useful property on path-connected topological spaces:

Theorem 2.8. *Let X be a path-connected topological space containing x_1 and x_2. Then $\pi_1(X, x_1) \cong \pi_1(X, x_2)$, where $\cong$ denotes group isomorphism.*

Intuitively, fundamental groups seem to be a local property of the basepoints of their loops. However, this theorem shows that in fact fundamental groups based at any point are isomorphic.

Proof Sketch. Consider the path $p : [0,1] \to X$ starting at x_1 and ending at x_2 (this path exists due to the path-connectedness requirement), and let $\bar{p}$ be the reverse. Then if γ is a loop based at x_1, then $p * \gamma * \bar{p}$ is a loop based at x_2. Thus, it remains to show that $\varphi : \pi_1(X, x_1) \to \pi_1(X, x_2)$ defined by $\varphi([\gamma]) = [p * \gamma * \bar{p}]$ is an isomorphism. Its inverse is given by $[\bar{p} * \gamma * p]$; proving that it is a homomorphism is a matter of definitions. ■

As a result of Theorem 2.8, we may simply refer to the fundamental group as $\pi_1(X)$, since the basepoint itself is irrelevant to the group structure. The notation will be used interchangeably throughout this paper, with $\pi_1(X, x)$ emphasizing $x \in X$.

It is beneficial to consider objects in relation to each other. So far, we have established a relationship between fundamental groups within a single topological space. The more general question arises: how are, say, $\pi_1(X)$ and $\pi_1(Y)$ related?

Theorem 2.9. *Let $f : X \to Y$ be a continuous map, and suppose $f(x) = y$ for some $x \in X, y \in Y$. Then $f_* : \pi_1(X, x) \to \pi_1(Y, y)$, defined by $f_*([\gamma]) := [f \circ \gamma]$, is the* induced homomorphism *of f.*

Proof Sketch. We must show that f_* is well-defined and that it is a homomorphism. For the former, we must show that if $\gamma \sim \gamma'$ in X, then $f \circ \gamma \sim f \circ \gamma'$ in Y. But this is easy: if H is a homotopy mapping γ to γ', then $f \circ H$ is a homotopy mapping $f \circ \gamma$ to $f \circ \gamma'$. For the latter, using the definition of $*$, it is straightforward to check that indeed $(f \circ \gamma) * (f \circ \gamma') = f \circ (\gamma * \gamma')$. ■

Lastly, we introduce the notion of simple-connectedness, which we will utilize later on in this paper.

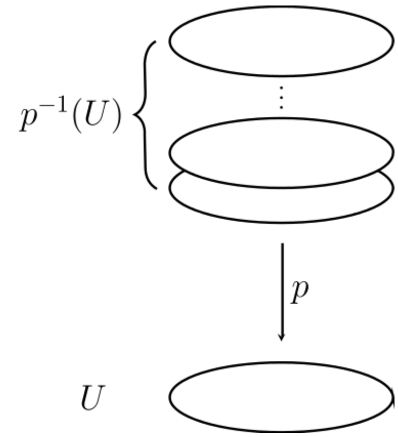


FIGURE 2. A visual representation of the covering map p, which surjective is locally a homeomorphism, as represented by the "stack of pancakes" atop an open set U.

Definition 2.10. A space is called *simply-connected* if it is path-connected and has trivial fundamental group.

The following result explains the name. Whereas homotopies must preserve endpoints, the changes in the basepoints explored in the previous results motivates a discussion on topological spaces in which the endpoints of paths are ignored by homotopy classes.

Proposition 2.11. *X is simply-connected if and only if there is a unique homotopy class of paths connecting any two points in X.*

Proof. For the first direction, suppose that X is simply-connected. If f and g are two paths with identical start and end points, then $f \sim f * \overline{g} * g \sim g$ since $\overline{g} * g$ and $f * \overline{g}$ are loops and are thus homotopic to the constant path.

For the other direction, note the special case that there is only one homotopy class of paths connecting a basepoint x to itself. Thus, $\pi_1(X, x) = 0$. ■

3. Covering Maps

We begin by introducing relevant terminology and discuss several examples. First we introduce the formal notion of a covering space.

Definition 3.1. Let X and $\widetilde{X}$ be topological spaces. Suppose $p : \widetilde{X} \to X$ is continuous. Then an open subset U of X is *evenly covered* if and only if $p^{-1}(U)$ is the disjoint union of open sets in $\widetilde{X}$, each of which mapping homeomorphically onto U by p. We say that p is a *covering map* if it is surjective and, moreover, every point in X is contained in an open neighborhood that is evenly covered. If p is a covering map, then we say that $\widetilde{X}$ is a *covering space* of X, the *base space*. See Figure 2.

The simplest example of a covering space is the trivial map, i.e. $\widetilde{X} = X$ and $p = \mathrm{id}$. Clearly, any open subset of X is mapped onto itself homeomorphically (we don't even have to choose a U for each $x \in X$; anything works!). This example does not provide any useful

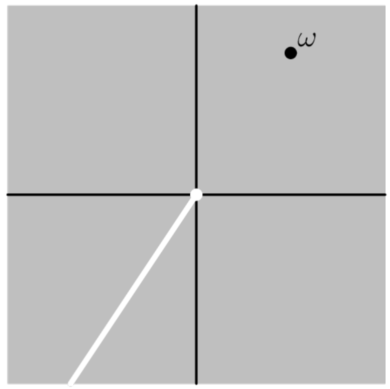


FIGURE 3. U corresponding to θ in Example 3.3

insights, though. The following example, which is still fairly simple, uses a covering space that is not homeomorphic to the base space.

Example 3.2. Define the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$ as usual. We may parameterize it by the continuous, surjective map $p : \mathbb{R} \to \mathbb{S}^1$,

$$p(t) = (\cos t, \sin t).$$

We claim that $p : \mathbb{R} \to \mathbb{S}^1$ is a covering map. Consider some point $x \in \mathbb{S}^1$, and take the open set containing $x = p(T)$ defined by $U = \mathbb{S}^1 - \{-x\}$. Then $p^{-1}(U)$ consists of several disjoint open intervals,

$$\bigcup_{n \in \mathbb{Z}} (T + (2n-1)\pi, T + (2n+1)\pi).$$

Each of these intervals is mapped homeomorphically onto U by p, and the claim follows. $\square$

Let us consider the slightly more complex example involving a homotope of $\mathbb{S}^1$, namely $\mathbb{C} - \{0\}$. We can use roughly the same approach by considering the image of $\mathbb{S}^1$ under the homotopy transforming it to $\mathbb{C} - \{0\}$.

Example 3.3. Consider the punctured plane $\mathbb{C} - \{0\}$. We claim that the $p : \mathbb{C} \to \mathbb{C} - \{0\}$ defined by $p(z) = e^z$ is a covering map. Note that p is surjective and continuous.

Analogously to the Example 3.2, fix some $\omega = Re^{i\theta} \in \mathbb{C} - \{0\}$ with $R > 0$ and $\theta \in \mathbb{R}$. Consider the open set containing $z \in p^{-1}(\omega)$ defined by $U = \mathbb{C} - \ell_\theta$, where $\ell_\theta = \left\{re^{i\theta} : r \leq 0\right\}$ is the ray beginning at the origin and pointing opposite the direction of ω (see Figure 3). Observe that every point in U can be uniquely written in the form $re^{i\varphi}$ with $r > 0$ and $\varphi \in (\theta - \pi, \theta + \pi)$.

This motivates us define $H_\alpha = \{z \in \mathbb{C} : \operatorname{Im}(z) \in (\alpha - \pi, \alpha + \pi)\}$, where Im denotes the imaginary part. Suppose $a + bi \in p^{-1}(U)$, i.e. $e^a e^{bi} \in U$. Any $a \in \mathbb{R}$ is allowed, as exp is a bijection $\mathbb{R} \mapsto (0, \infty)$. Any $b \not\equiv \theta + \pi \pmod{2\pi}$ is allowed by the description of U above. Thus, it follows that

$$p^{-1}(U) = \bigcup_{n \in \mathbb{Z}} H_{\theta + 2\pi n}.$$

It remains to prove that U is evenly covered by p, but these observations make the task fairly straightforward. By the definition of H, the $H_{\theta+2\pi n}$'s are pairwise disjoint. (Compare this to the exposition preceding this example: if we shrink along the Re axis, we see that this is homotopic to the union of open intervals along the real line.) Since p is continuous on its full domain, it is also continuous over its restriction to H_α. Thus p is a covering map, as desired. □

Example 3.4. If $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ are covering maps, then so is $p \times q : \widetilde{X} \times \widetilde{Y} \to X \times Y$, given by

$$(p \times q)(\widetilde{x}, \widetilde{y}) = (p(\widetilde{x}), q(\widetilde{y})).$$

For any point in the product, we can take the open set defined by the product of the corresponding open sets in X and Y. □

For reference, here follows a non-example.

Non-example 3.5. We claim that the map $p : (-2\pi, 2\pi) \to \mathbb{S}^1$ defined by $p(t) := (\cos t, \sin t)$ is *not* a covering map. Specifically, there is no open set U containing the point $(1, 0)$ that is evenly covered by p. If there were such a U, then choose ε sufficiently small that $U_\varepsilon = \{(\cos t, \sin t) : -\varepsilon < t < \varepsilon\} \subset U$. Then U_ε must be evenly covered by p, yet $p^{-1}(U_\varepsilon) = (-2\pi, -2\pi + \varepsilon) \cup (-\varepsilon, \varepsilon) \cup (2\pi - \varepsilon, 2\pi)$. But neither the first nor the last interval is mapped homeomorphically onto U_ε by p, so p cannot be a covering map. □

Having discussed several examples of covering maps, we are ready to begin a more general discussion of results. Given that the definition of continuity has the "if" and "then" clauses reversed, the following property is somewhat surprising:

Proposition 3.6. *Let $p : \widetilde{X} \to X$ be a covering map. Then $p(V)$ is open in X for every open set V in $\widetilde{X}$.*

Proof. Let $V \subset \widetilde{X}$ be open, and select some $v \in V$. Since p is a covering map, there exists an open set $U \subset X$ containing $x = p(v)$. Since $p^{-1}(U)$ is the disjoint union of open sets, each of which homeomorphically mapped by p onto U. At least one of these open sets must contain v, say $\widetilde{U}$; let $N_x = p(V \cap \widetilde{U})$. Since p is a homeomorphism and $V \cap \widetilde{U}$ is open, it follows that $N_x \subset p(V)$. In particular, by surjectivity,

$$p(V) = \bigcup_x N_x$$

and $p(V)$ is therefore itself open. ■

In particular, observe the following corollary:

Corollary 3.7. *A bijective covering map is a homeomorphism.*

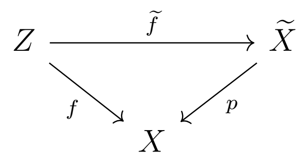
Proof. Proposition 3.6 demonstrates that p^{-1} is open; we know by the definition of p that it is continuous and a bijection. ■

4. Lifts into Covering Spaces

Thus far, we have explored covering maps as a standalone property. Here, we consider maps from a separate topological space to the base space and covering space and the relationships between these maps.

Definition 4.1. Let $p : \widetilde{X} \to X$ be a covering map, let Z be a topological space, and let $f : Z \to X$ be a continuous map. A continuous map $\widetilde{f} : Z \to \widetilde{X}$ is said to be a *lift* of f to the covering space $\widetilde{X}$ if $p \circ \widetilde{f} = f$.

Visually, this definition means that the following diagram is commutative; intuitively, it means that we can "lift" f past p.



The lift has several notable topological properties, which we will discuss throughout this paper. In particular, we prove that any lift of a covering map is uniquely determined by its value at a single point in its domain.

Proposition 4.2. *Let $p : \widetilde{X} \to X$ be a covering map, let Z be a connected topological space, and let $g, h : Z \to \widetilde{X}$ be continuous. If $p \circ g = p \circ h$ and $g(z) = h(z)$ for some point $z \in Z$, then $g = h$.*

Proof. Since Z is connected, then its only clopen sets are Z and $\varnothing$ (otherwise, if $\varnothing \subset Y \subset Z$, then $Y \sqcup (Z - Y)$ partition Z into disjoint open sets). The converse is also true. We are given that $Z_0 = \{z \in Z : g(z) = h(z)\}$ is nonempty, so proving that it is clopen would yield $Z_0 = Z$.

First, we define some notation. Let $z \in Z$. Let $U \subset X$ be evenly covered with $p(g(z)) \in U$. Then $p^{-1}(U)$ consists of the disjoint union of open sets, one of which, $\widetilde{U}$, contains $g(z)$. Likewise define V and $\widetilde{V}$ based on $h(z)$. Then $N_z = g^{-1}(\widetilde{U}) \cap h^{-1}(\widetilde{V})$ is an open subset of Z containing z. We split the claim into two cases:

Consider $z \in Z_0$. Then $g(z) = h(z) \implies \widetilde{U} = \widetilde{V}$, so both g and h map N_z into $\widetilde{U}$. Since $p|_{\widetilde{U}}$ is a homeomorphism, it follows that $g|_{N_z} = h|_{N_z}$, and $N_z \subset Z_0$. Thus z, which was selected arbitrarily, has an open neighborhood; Z_0 is open.

Finally, consider $z \notin Z_0$. Then $g(z) \neq h(z) \implies \widetilde{U} \cap \widetilde{V} = \varnothing$. Thus $g(N_z) \subset \widetilde{U}$ and $h(N_z) \subset \widetilde{V}$ yields $N_z \subset Z - Z_0$; similar to the previous case, we conclude that $Z - Z_0$ is open, and hence Z_0 is closed.

Thus Z_0 is clopen, as desired. ■

This fact that a lift is determined by only one point will be used several times throughout this paper. Notably, we will apply it to continuous paths.

Yet proving uniqueness of lifts is insufficient, as we have not yet proven that they exist. As we saw in §3, devising a meaningful lift can be challenging; we cannot generalize our strategies from $\mathbb{S}^1$ and $\mathbb{C} - \{0\}$ to other topological spaces that may not be homotopic or homeomorphic to them; even constructing those lifts was not entirely trivial. The following theorem demonstrates that lifts necessarily exist for paths, but does not explicitly construct the lift.

Theorem 4.3 (Path-Lifting Theorem). *Let $p : \widetilde{X} \to X$ be a covering map over a topological space X. Let $\gamma : [0,1] \to X$ be a continuous path, and suppose $\omega \in \widetilde{X}$ such that $p(\omega) = \gamma(0)$. Then there exists a unique continuous map $\widetilde{\gamma} : [0,1] \to \widetilde{X}$ for which $\widetilde{\gamma}(0) = \omega$ and $p \circ \widetilde{\gamma} = \gamma$.*

Proof. For $s \in [0,1]$, let $\eta_s : [0,s] \to X$ be a continuous map such that $\eta_s(0) = \omega$ and $p(\eta_s(t)) = \gamma(t)$ for $t \in [0,1]$, assuming such a map exists. If so, write $s \in S$, and define $M = \sup S$ (i.e. the least upper bound). We claim that $M = 1$, in which case taking $\widetilde{\gamma} = \eta_1$ suffices.

Since p is a covering map, there exists some evenly covered open neighborhood U of $\gamma(M)$. By continuity, we may select some open neighborhood $V \subset [0,1]$ of M such that $\gamma(V) \subset U$. Choose some element $s \in S \cap V$, so η_s exists. Then $p^{-1}(U)$ is the disjoint union of open sets, one of which containing $\eta_s(s)$, say $\widetilde{U}$. Thus we may define the unique continuous map $q : U \to \widetilde{U}$ such that $p(q(x)) \equiv x$; in particular, $q(\gamma(s)) = \eta_s(s)$.

Given this relation, observe that the following function is continuous, where $s \in V$:

$$\zeta_s = \begin{cases} \eta_M(t) & t \in [0, M] \\ q(\gamma(t)) & t \in [M, s]. \end{cases}$$

Thus $s \in S \implies V \subset S$, which can only occur if $M = 1$, and η_1 suffices. Uniqueness follows from Proposition 4.2. ■

Moreover, we claim that homotopies also have a unique lifting. As we will see in Proposition 4.5, combining the theorems for path-lifting and homotopy-lifting yields an important result.

Theorem 4.4 (Homotopy-Lifting Theorem). *Let $p : \widetilde{X} \to X$ a covering map. Let Z be a topological space, and let $F : Z \times [0,1] \to X$ and $g : Z \to \widetilde{X}$ be continuous maps with the property that $p \circ g(z) \equiv F(z,0)$. Then there exists a unique continuous map $G : Z \times [0,1] \to \widetilde{X}$ such that $G(z,0) \equiv g(z)$ and $p \circ G = F$.*

The full proof is long and tedious; we have omitted it here, but the interested reader may see [Wil17] or [MCJ77].

Proof Sketch. Consider the path $\gamma_z : [0,1] \to Z$ defined by $\gamma_z(t) \equiv F(z,t)$ for each $z \in Z$. Then, utilize Theorem 4.3 to lift each of these γ_z to some $\widetilde{\gamma}_z$ so that $p \circ \widetilde{\gamma}_z = \gamma_z$. Then, define $G : [0,1] \times Z \to \widetilde{X}$ by $G(t,z) = \widetilde{\gamma}_z(t)$. It is straightforward to check the elementary properties; it remains to prove that G is continuous and unique.

For continuity, we use the same idea as in Theorem 4.3. In other words, define a set S_z for each $z \in Z$ analogously to our old definition of S. The goal, as before, is to show that a neighborhood of $\sup S_z$ is contained in S_z, which would give $\sup S_z = 1$. It is lengthy, but not particularly difficult, to prove using the same argument on open sets.

Uniqueness follows directly from the uniqueness of $\widetilde{\gamma}_z$ satisfying the necessary properties. ■

As previously alluded to, we combine the last two theorems to yield the following result.

Proposition 4.5. *Let $p : \widetilde{X} \to X$ be a covering map over X, let $\alpha, \beta : [0,1] \to X$. Suppose that $\widetilde{\alpha}(0) = \widetilde{\beta}(0)$ and $\alpha \sim \beta \operatorname{rel}\{0,1\}$. Then $\widetilde{\alpha} \sim \widetilde{\beta} \operatorname{rel}\{0,1\}$.*

Note the requirement that $\widetilde{\alpha}(0) = \widetilde{\beta}(0)$. This additional condition is necessitated by Proposition 4.2.

Proof. Let $x_0 = \alpha(0) = \beta(0)$ and $x_1 = \alpha(1) = \beta(1)$, and let $F : [0,1] \times [0,1] \to x$ be the homotopy between α and β satisfying $F(0,t) \equiv \alpha(t)$ and $F(1,t) \equiv \beta(t)$. By Theorem 4.4, there exists a $G : [0,1] \times [0,1] \to \widetilde{X}$ such that $p \circ G = F$ and $G(0,0) = \widetilde{x_0}$.

Thus it remains to check that G serves its desired function, mapping $\widetilde{\alpha}$ to $\widetilde{\beta}$ and satisfies there$\operatorname{rel}\{0,1\}$ condition. Since

$$p(G(0,t)) \equiv x_0 \quad \text{and} \quad p(G(1,t)) \equiv x_1 \quad \text{for all } t \in [0,1],$$

it follows from Proposition 4.2 (the lift of a constant path must be a constant path) that $G(0,t) = \widetilde{x_0}$ and $G(1,t) = \widetilde{x_1}$ for all $t \in [0,1]$, where $\widetilde{x}_0 = G(0,0) = \widetilde{\alpha}(0)$ and $\widetilde{x}_1 = G(1,0) = \widetilde{\beta}(0)$.

Observe that

$$p(G(t,0)) = F(t,0) = \alpha(t) = p(\widetilde{\alpha}(t)) \quad \text{and}$$
$$p(G(t,1)) = F(t,1) = \beta(t) = p(\widetilde{\beta}(t)).$$

Again by Proposition 4.2, since $\widetilde{\alpha}$ and $\widetilde{\beta}$ are uniquely defined, we must have $G(t,0) = \widetilde{\alpha}(t)$ and $G(t,1) = \widetilde{\beta}(t)$. In particular, $\widetilde{\alpha}(1) = G(1,0) = \widetilde{x_1} = G(1,1) = \widetilde{\beta}(1)$. Thus, G is the homotopy that yields the desired condition, $\widetilde{\alpha} \sim \widetilde{\beta} \operatorname{rel}\{0,1\}$. ■

We conclude this section with a discussion of useful results regarding the fundamental group. The covering map provides insights into relationships between the spaces themselves. The following two results are independent, but they both show strong similarities to the properties of covering maps and spaces we have already discussed.

Proposition 4.6. *Let $p : \widetilde{X} \to X$ be a covering map, and let $\widetilde{x} \in \widetilde{X}$. Then*

$$p_* : \pi_1(\widetilde{X}, \widetilde{x}) \to \pi_1(X, p(\widetilde{x}))$$

is injective.

Proof. Let $[\sigma], [\tau] \in \pi_1(\widetilde{X}, \widetilde{x})$. Suppose $p_*([\sigma]) = p_*([\tau])$. Since $p \circ \sigma \sim p \circ \tau$ with identical starting and ending points and $\sigma(0) = \widetilde{x} = \tau(0)$, it follows from Proposition 4.5 that $\sigma \sim \tau$ with identical start and end points. Thus p_* is injective. ■

Theorem 4.7. *Let $p : \widetilde{X} \to X$ be a covering map over a topological space X. If $\widetilde{X}$ is path-connected and X is simply connected (i.e. any two paths with the same start and end points are homotopic), then p is a homeomorphism.*

Such a statement is rather surprising — only two restrictions are needed on the involved topological spaces to impose such a tight constraint on p. Before we prove this theorem, we will first prove the following lemma.

Lemma 4.8. *Let $p : \widetilde{X} \to X$ be a covering map. Suppose $\widetilde{X}$ is path-connected, and let $\omega_0, \omega_1 \in \widetilde{X}$ such that $p(\omega_0) = p(\omega_1)$. If path $\alpha : [0,1] \to \widetilde{X}$ starts at ω_0 and ends at ω_1 and $[p \circ \alpha] \in p_*(\pi_1(\widetilde{X}))$, then α is a loop, i.e. $\omega_0 = \omega_1$.*

Proof Sketch. First, observe that if γ is a loop in X with $[\gamma] \in p_*(\pi_1(\widetilde{X}))$, then there exists a loop $\widetilde{\gamma}$ in $\widetilde{X}$ such that $p \circ \widetilde{\gamma} = \gamma$. As an outline of the proof, suppose that $[\gamma] = p_*([\sigma])$; then $\gamma \sim p \circ \sigma \operatorname{rel} \{0, 1\}$, after which point Theorem 4.3 and Proposition 4.5 do the job. We leave it to the reader to fill in the details of the proof.

Thus, we have $p \circ \beta = p \circ \alpha$ for some loop β based at ω_0. Since $\alpha(0) = \beta(0)$, it follows from Proposition 4.2 that $\alpha = \beta$ as desired. ∎

Proof of Theorem 4.7. By Corollary 3.7, it suffices to show that p is a bijection. Since we are given that p is a covering map and hence surjective, it remains to show that it is injective. Suppose that $\omega_0, \omega_1 \in \widetilde{X}$ satisfying $p(\omega_0) = p(\omega_1)$. By path-connectedness, we may define a path $\alpha : [0, 1] \to \widetilde{X}$ starting at ω_0 and ending at ω_1; in particular, $p \circ \alpha$ is a loop in X. By Lemma 4.8, it follows that $\omega_0 = \omega_1$, so p is injective as desired. ∎

Having explored paths and homotopies, also important to know about the existence of lifts of general maps. (We've already seen uniqueness in Proposition 4.2.) The proof of this proof is rather long and technical, so it has been omitted. However, since this result is of particular significance, we encourage the reader to see the full proof [HPoM02].

Theorem 4.9 (Lifting Criterion)**.** *Suppose $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space, and $f : (Y, y_0) \to (X, x_0)$ is a map with Y path-connected and locally path-connected. Then a lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Note. In general, properties are true *locally* if each point has arbitrarily small neighborhoods with that property, i.e. for every point $x \in X$ and for every $U \subset X$ containing x, there exists a $V \subset U$ containing x and satisfying the property. They are true *semilocally* if for each $x \in X$, there exists a neighborhood U of x satisfying the property — a weaker condition.

5. Universal Cover

Thus far, we have discussed relationships involving topological spaces, the fundamental group, and covering spaces. In this section, we will construct a covering space for a given topological space X, called the *universal cover*. In particular, we provide certain constraints such that this cover is uniquely determined based on X alone.

Definition 5.1. A topological space is called *semilocally simply-connected* if for each point in $x \in X$, there exists a neighborhood U of x such that the inclusion-induced map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

We claim that the following conditions are sufficient and necessary to provide a unique simply-connected covering space:

Theorem 5.2. *Suppose X is path-connected, locally path-connected, and semilocally simply-connected. We can directly construct a simply-connected covering space of X.*

Let us motivate the construction of $\widetilde{X}$. Suppose $p : \widetilde{X} \to X$ is a simply-connected covering space. By Proposition 2.11, each $\widetilde{x} \in \widetilde{X}$ can be joined to $\widetilde{x}_0$ by a unique homotopy class of paths in X starting at x_0. With this in mind, we intended to define $\widetilde{X}$ purely in terms of X. We use the natural choice:

$$\widetilde{X} = \{[\gamma] : \gamma \text{ is a path in } X \text{ starting at } x_0, \text{ ending at } x_1\}.$$

We define $p : \widetilde{X} \to X$ by $[\gamma] \to x_1$; note that x_1 can be any point in X due to path-connectedness, so p is surjective.

We aim to characterize $\widetilde{X}$ rather than simply describing it and stating that it exists. In particular, let us describe a topology on $\widetilde{X}$.

Proposition 5.3. *Let $\mathcal{U}$ be the collection of path-connected open sets $U \subset X$ with the inclusion-induced map $\pi_1(U) \to \pi_1(X)$ trivial. Then $\mathcal{U}$ is a topology on $\widetilde{X}$.*

Proof. Given $U \in \mathcal{U}$ and a path γ in X from x_0 to a point in U, let

$$U_{[\gamma]} = \{[\gamma \cdot \eta] : \eta \text{ is a path in } U \text{ starting at } x_1\}.$$

We claim that $U_{[\gamma]}$ forms a base for a topology on $\widetilde{X}$.

- Any path-connected open subset $V \subset U \in \mathcal{U}$ is alson in $\mathcal{U}$, as the composition $\pi_1(V) \to \pi_1(U) \to \pi_1(X)$ is also trivial.
- $p : U_{[\gamma]} \to U$ is surjective, since different choices of η connecting x_1 to a fixed $x \in U$ are all homotopic in X. Thus, $U_{[\gamma]}$ covers U.
- $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$. Suppose $\gamma' = \gamma * \eta$. Then the elements of $U_{[\gamma']}$, of the form $[\gamma * \eta * \mu]$, are contained in $U_{[\gamma]}$. Likewise, elements of $U_{[\gamma]}$ have the form $[\gamma * \mu] = [\gamma * \eta * \overline{\eta} * \mu] = [\gamma' * \overline{\eta} * \mu]$ and are contained in $U_{[\gamma']}$.
  In particular, let us consider some $\gamma'' \in U_{[\gamma]} \cap U_{[\gamma']}$. By the above, we have $U_{[\gamma]} = U_{[\gamma']} = U_{[\gamma'']}$, so $U_{[\gamma'']}$ meets the requirements.

The required properties are thus satisfied. ■

Finally, it remains to prove that $\widetilde{X}$ is simply connected to complete the proof of Theorem 5.2.

Proposition 5.4. *$\widetilde{X}$ is simply connected.*

Proof. Fix $[\gamma] \in \widetilde{X}$, and consider its lifting in $\widetilde{X}$ that starts at $[x_0]$ (the homotopy class of the constant path x_0) and ends at $[\gamma]$. Hence $\widetilde{X}$ is path-connected; to show that it has trivial fundamental group, we can show that the image under the injection p_* is trivial. But this is clear: γ lifts to a loop starting at $[x_0]$, so γ is homotopic to the constant path as desired. ■

Thus we have constructed our desired covering space — the universal cover. We discuss several important results regarding the universal cover. First, we begin with constructing covering spaces for arbitrary subgroups of $\pi_1(X)$. This property, that the constructed covering space for X can be induced on any subgroup, is rather surprising.

Proposition 5.5. *Suppose X is path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \subset \pi_1(X, x_0)$, there is a covering space given by $p : X_H \to X$ such that $p_*(\pi_1(X,\widetilde{x}_0)) = H$ for a suitable basepoint $\widetilde{x}_0 \in X_H$.*

Proof. For $[\gamma], [\gamma']$ in the constructed covering space $\widetilde{X}$, define $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma * \overline{\gamma'}] = H$; one may easily check that this is an equivalence relation. Let $X_H = H/\sim$. We claim that the projection $X_H \to X$ induced by $[\gamma] \mapsto \gamma(1)$ is a covering space.

Let the basepoint for $\widetilde{x}_0 \in X_H$ be $[x_0]$; we claim that the image of the map $p_* : \pi_1(X_H, \widetilde{x}_0) \to \pi_1(X, x_0)$ is H. For a given loop γ in X based at x_0, its lift to $\widetilde{x}$ starts at $[x_0]$ and ends at $[\gamma]$. This is a loop if and only if $[\gamma] \sim [x_0] \iff [\gamma] \in H$. ■

Finally, we answer the question: is *the* universal cover actually unique? In §3, we explored numerous concrete examples of direct constructions of covering spaces without even considering the fundamental group. The following theorem answers this question subject to the additional requirements of path-connectedness, local path-connectedness, and semilocal simply-connectedness. As with group theory, we are interested in uniqueness up to isomorphism. Let us define this.

Definition 5.6. Let $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ correspond to covering spaces of X. A *homomorphism* $(\widetilde{X}_1, p_1) \to (\widetilde{X}_2, p_2)$ is a continuous map $\varphi : \widetilde{X}_1 \to X_2$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X_1} & \xrightarrow{\varphi} & \widetilde{X_2} \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & X & \end{array}$$

Furthermore, $\varphi : (\widetilde{X}_1, p_1) \to (\widetilde{X}_2, p_2)$ is an *isomorphism* if there exists an inverse homomorphism $\psi : (\widetilde{X}_2, p_2) \to (\widetilde{X}_1, p_1)$.

Thus the covering space structures are preserved. Notably, isomorphisms are actually homeomorphisms by Corollary 3.7. The inverse of an isomorphism is an isomorphism, as is the composition of two isomorphisms. Thus it is an equivalence relation. Now, we are ready to state our theorem.

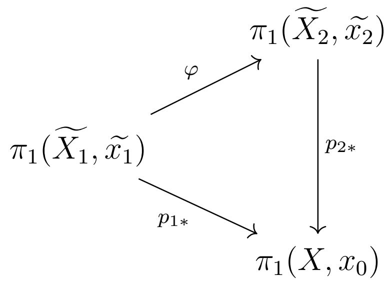
Theorem 5.7. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$ are bijective.*

Moreover, the correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : \widetilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

First we prove the following lemma, which nearly gives us the first half of the theorem.

Lemma 5.8. *If X is path-connected and locally path-connected, then two path-connected covering spaces given by $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ are isomorphic via $f : \widetilde{X}_1 \to \widetilde{X}_2$ where $\widetilde{x}_1 \in p_1^{-1}(x_0)$ is taken to a basepoint $\widetilde{x}_2 \in p_2^{-1}(x_0)$ if and only if p_{1*}.*

Proof Sketch. Apply Theorem 4.9 in two directions.



■

Proof of Theorem 5.7. The first part is obtained from Lemma 5.8 by associating the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ with the covering space $(\widetilde{X}, x_0)$.

For the second part, we show that for a given $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, changing the basepoint $\widetilde{x}_0$ within $p^{-1}(x_0)$ corresponds to changing $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ to a conjugate subgroup of $\pi_1(X, x_0)$. Specifically, choose $\widetilde{x}_1 \in p^{-1}(x_0)$, and let $\widetilde{\gamma}$ be a path from $\widetilde{x}_0$ to $\widetilde{x}_1$. Let $H_i = p_*(\pi_1(\widetilde{X}, \widetilde{x}_i))$ for $i = 0, 1$. For loops $\widetilde{f}$ at $\widetilde{x}_0$, $\overline{\widetilde{\gamma}} * \widetilde{f} * \widetilde{\gamma}$ is a loop at $\widetilde{x}_1$, so $g^{-1}H_0 g \subset H_1$. Likewise, $gH_1g^{-1} \subset H_0$, from which it follows that $H_1 = g^{-1}H_0 g$ as desired. ■

6. Conclusion

Theorems 5.5 and 5.7 bear a striking similarity to the following result from the purely algebraic subject Galois theory, with subgroups analogous to fields. For this reason, the correspondence discussed in its proof is frequently called the "Galois correspondence" despite its use in geometry.

Theorem (Part of the Fundamental Theorem of Galois Theory)**.** *Let L/F be a finite Galois extension. There is a bijection between intermediate fields K between L and F, and subgroups of* $\text{Gal}(L/F)$*. The bijection is given by* $\Phi : K \to \text{Gal}(L/K)$.

The question is raised: does a larger result encapsulate both of these "intermediate" bijections? Can problems from one theory be solved with methods from the other?

Acknowledgements. The ideas, theorems, and proofs presented in this paper are the outgrowth of an intensive study of Algebraic Topology at the Stanford University Mathematics Camp (SUMaC). I am very grateful to my instructor, Maxim Gilula, for providing the background and inspiration for this paper.

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